SNAP 2017. Laplace's equation and conformal maps.

Problem Set 1

1. Consider the holomorphic map $w = z^2$. For $a, b > 0$, what are the images of the hyperbolae
$$x^2 - y^2 = a, \quad 2xy = b$$
under this map? (We are writing $z = x + iy$).

2. Let a, b be nonzero real numbers. Show that the map $w = 1/z$ transforms the vertical line $x = a$ to a circle through the origin centered at $(\frac{1}{2a}, 0)$. What about horizontal lines $y = b$?

3. Recall that the holomorphic function $\sin z$ is defined by $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$. Recall also that $e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$.
   (a) Show that $\sin z = \sin x \cosh y + i \cos x \sinh y$, where we recall that hyperbolic sine and cosine are defined by
   $$\sinh y = \frac{e^y - e^{-y}}{2}, \quad \cosh y = \frac{e^y + e^{-y}}{2}.$$
   (b) For $a, b \neq 0$, show that the lines $x = a$ and $y = b$ are transformed into hyperbolae and ellipses respectively, and find their equations. (Hint: use the identities $\cosh^2 x - \sinh^2 x \equiv 1$ and $\sin^2 x + \cos^2 x \equiv 1$.)

4. Find a conformal map from the upper half plane $\{y > 0\}$ onto $\{w \in \mathbb{C} \mid |w| > 1\}$.

5. Find the Möbius transformation $f(z) = \dfrac{az + b}{cz + d}$ which maps 1 to 1, -1 to -1 and i to 0.

6. Without computing the map explicitly, find the image of the open unit disk $\{|z| < 1\}$ under the Möbius transformation that maps -1 to $-i$, 1 to $2i$ and i to 0.

7. (a) Find a holomorphic surjection from the upper half plane onto $\mathbb{C}$.
   (b) Why is there no holomorphic surjection from $\mathbb{C}$ onto the upper half plane? *(Hint: Liouville's Theorem says that a bounded holomorphic map on $\mathbb{C}$ must be constant.)*

8. Consider the Joukowsky map $f(z) = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$.

   (a) Where is $f(z)$ conformal? (Recall that a map $g(z)$ is *conformal* at z if it is holomorphic at z and $g'(z) \neq 0$.)

   (b) Show that for $w \in \mathbb{C}$, the equation $f(z) = w$ has exactly two solutions z except when $w = \pm 1$.

   (c) Show that $w = f(z)$ is a bijective conformal map from the half disk $\{|z| < 1,\ \operatorname{Im} z > 0\}$ to the lower half plane $\{w \in \mathbb{C} \mid \operatorname{Im} w < 0\}$.

9. Using the Joukowsky map, composed with other conformal maps, find the image of the semi-infinite vertical strip $S = \{(x, y) \mid |x| < \pi/2,\ y > 0\}$ under the map $w = \sin z$. What are the images of the horizontal line segments $y = c$ inside S under the map?